ISRAEL JOURNAL OF MATHEMATICS **135** (2003), 221–250

# POSSIBLE ENTROPY FUNCTIONS

BY

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ABSTRACT

We characterize the class of functions which occur as the entropy function defined on the set of invariant measures of a (minimal) topological dynamical system. Namely, these are all non-negative affine functions $h$, defined on metrizable Choquet simplices, which are non-decreasing limits of upper semi-continuous functions. If $h$ is itself upper semi-continuous then it can be realized as the entropy function in an expansive dynamical system.The constructions are done effectively using minimal almost 1-1 extensions over a rotation of a group of $p$-adic integers (in the expansive case, the construction leads to Toeplitz flows).

## Introduction

By a topological dynamical system $(X,T)$ we shall mean a compact metric space $X$ with a continuous map $T\colon X \mapsto X$. The set of all invariant measures of such system is well-known to be a non-empty compact metrizable Choquet simplex. Recall that a convex compact subset of a metric locally convex linear space is a Choquet simplex if each of its points is a barycenter of a unique probability measure supported by the extreme points (see [P1] for more information on simplices). In [D1] it is proved that every such simplex can occur (up to affine homeomorphism) as the set of invariant measures already within the class of minimal expansive systems.

* Supported by the KBN grant 2 P03 A 04622 [02-05 r.].
Received July 13, 2001

Let us now consider a pair $(K, h)$, where $K$ is a metrizable Choquet simplex and $h$ is an affine function on $K$ taking values in the extended interval $[0, \infty]$. We say that two such pairs $(K, h)$ and $(K', h')$ are **isomorphic** if there exists an affine homeomorphism $\pi$ between Choquet simplices $K$ and $K'$ such that $h' \circ \pi \equiv h$; from now on we identify isomorphic pairs $(K, h)$.

In this note we will investigate possible forms of the pair $(K, h)$, where $K = K(X, T)$ is the simplex of all $T$-invariant Borel probability measures on $X$ and $h$: $K \mapsto [0, \infty]$ is the entropy function assigning to each invariant measure its entropy with respect to $T$. This pair is obviously an invariant of topological conjugacy, and it has been a subject of study in many contexts (see [P2] for a recent application). It is natural to ask the following questions:

(1) Which pairs $(K, h)$ are admitted in topological dynamical systems?
(2) Which pairs $(K, h)$ are admitted in expansive dynamical systems?
(3) Which pairs $(K, h)$ are admitted in minimal dynamical systems?

In this work we will provide complete answers to questions (1)–(3). Namely, apart from the arbitrary choice of $K$ (as shown in [D1]), the function $h$ can also be arbitrary in the following classes:

- $AU_1^+(K)$ of all affine upper semi-continuous (u.s.c.) functions on $K$ into $[0, \infty)$ (hence bounded), in the case (2);
- $AL_2^+(K)$ of all affine functions $h$: $K \mapsto [0, \infty]$ which are non-decreasing limits of sequences of u.s.c. functions, in the case (1);
- minimality assumption (3) imposes no additional restrictions.

We remark that the class $L_2$ of all non-decreasing limits of sequences of u.s.c. functions is contained (strictly, on sufficiently rich spaces) between the first and second Baire class (see, e.g., [K]).

Some parts of the statements displayed above are obvious: the entropy function $h$ is always non-negative and affine. It is also well-known that this function is bounded and u.s.c. in expansive (even in so-called asymptotically $h$-expansive) dynamical systems (see, e.g., [Mis]). The statement concerning minimal systems also requires almost no effort: we have at our disposal a theorem, which can be directly applied here. This is namely Theorem 3 in [D-L], which is a strengthening of the Furstenberg–Weiss theorem about replacing an arbitrary extension of a minimal system by a minimal almost 1-1 extension. Below we state that theorem in a form suitable for our purposes:

Fact 1 ([D-L], Theorem 3): *Let $(X, T_X)$ be an arbitrary dynamical system and let $(Z, T_Z)$ be a strictly ergodic non-periodic factor of $(X, T_X)$. Then there exists a minimal dynamical system $(Y, T_Y)$ which is an almost 1-1 extension of $(Z, T_Z)$,* *and which is Borel* isomorphic to* $(X, T_X)$. *If* $(X, T_X)$ *is expansive, so is* $(Y, T_Y)$. ∎

This theorem will apply because all dynamical systems which we construct will be extensions of a fixed odometer (hence of a strictly ergodic non-periodic dynamical system). We also remark that all these systems will be invertible, i.e., actions of homeomorphisms. As a result, the solution will be obtained within the class of Toeplitz flows (classical — for the expansive case, and in a slightly extended meaning — in general). It is not necessary to quote the definition of Borel* isomorphism (see [D-L]); it suffices to know that, although weaker than topological conjugacy, it preserves the pair $(K, h)$.

What remains to do is:

(a) to show that the entropy function is always in $AL_2^+$;

(b) to construct suitable dynamical systems: an expansive one for an arbitrary entropy function $h \in AU_1^+(K)$ on an arbitrary simplex $K$, and a non-expansive one for $h \in AL_2^+(K)$.

The fact (a) can be derived from the "operator approach" to entropy represented in several works (see, e.g., [Mak], [G-L-W]). Namely, entropy is defined in terms of measurable partitions of unity (see Definition 4 in [G-L-W]), and, for a continuous partition of unity $\Phi$, the entropy $h(\Phi)$ (denoted as $\epsilon({}^T\mathcal{D}, \Phi)$ in [G-L-W]) is immediately seen to be a u.s.c. function of the measure (in case of an action of a continuous transformation). The entropy of an operator is defined as the supremum over all measurable partitions of unity of the above entropies, and, for an operator induced by a transformation, it coincides with the classical notion of entropy (Theorem in [G-L-W]). Finally, for a continuous transformation, using a standard approximation argument, it is easy to select a sequence of continuous partitions of unity along which this supremum is attained universally for all invariant measures. This implies (a).

Obviously, (b) constitutes the main effort of this work. In the construction we will be using combinatorial methods like in [D1]. The result of [D1] was later re-proved using very elegant and powerful tools of strong orbit equivalence and the theory of dimension group (see Gjerde and Johansen [G-J] or Ormes [O]). It would be very interesting to adapt a dimension-group-theoretic argument to prove results similar to ours; however, since strong orbit equivalence does not preserve entropy, there seems to be no immediate way to do it. Although the framework of proofs is the same as in [D1], the level of complication is now much higher (all dynamical systems in [D1] have entropy zero). On the other hand, we can now ignore some technical details responsible for minimality of the constructed systems (e.g., the metric $d^{**}$). In any case, we shall provide all necessary details, so that parallel tracing of the old paper will not be necessary.

## Preliminaries

INVARIANT MEASURES IN SUBSHIFTS. As mentioned in the Introduction, a dynamical system is a pair $(X,T)$ where $X$ is a compact metric space and $T\colon X \mapsto X$ is continuous. It is known that in such a case the set $K(X,T)$ of all invariant (probability) measures on $X$ is nonempty and convex, and endowed with the weak* topology constitutes a compact metric Choquet simplex. The extreme points of $K(X,T)$ are exactly the ergodic measures on $X$.

By a **subshift over a finite alphabet** we will mean a dynamical system $(X,\sigma)$ where $X$ is a closed shift-invariant subset of $\Sigma^{\mathbb{Z}}$ ($\Sigma$ denotes some finite set called *alphabet* and is always regarded as a discrete topological space), and where the shift map is given by $\sigma(\omega)(n) := \omega(n+1)$ ($\omega = (\omega(n))_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$). By a block we will mean a finite sequence over $\Sigma$: $B \in \Sigma^n$ ($n \in \mathbb{N}$). The collection of all blocks $B$ of length $n$ appearing in (some element of) $X$ will be denoted by $\mathcal{B}_n(X)$, while $\mathcal{B}_n$ is a synonym for $\Sigma^n$. Expressions of the form $BC\cdots D$ with two or more blocks will denote their concatenation.

For subshifts, the weak* topology in the set of invariant measures coincides with one induced by the following metric:

$$\operatorname{dist}(\mu,\nu) := \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{B\in\mathcal{B}_n} |\mu(U_B) - \nu(U_B)|,$$

where $U_B$ denotes the cylinder $\{\omega \in \Sigma^{\mathbb{Z}} : \omega[0,n-1] = B\}$. This metric is easily seen to be convex (the balls are convex).

If $C$ is a block, then by $\mu_C$ we will denote the (unique) invariant measure carried by the (periodic) orbit of the sequence obtained by periodic repetitions of $C$. We will write $\operatorname{dist}(\mu, C)$ instead of $\operatorname{dist}(\mu,\mu_C)$. According to this convention we formulate below several well-known facts concerning approximating invariant measures by blocks:

FACT 2: *If $(X,\sigma)$ is a subshift over a finite alphabet then:*

(1) $(\forall\epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \ge n_0)$ $\mathcal{B}_n(X) \subset K^\epsilon(X,\sigma)$ *($K^\epsilon$ denotes the $\epsilon$-neighborhood of the set $K$);*

(2) *every ergodic measure on $X$ is a limit of a sequence of blocks (with lengths increasing to infinity) appearing in $X$;*

(3) $(\forall\epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \ge n_0)(\forall k \in \mathbb{N})\operatorname{dist}(\frac{1}{k}\sum_{i=1}^{k}\mu_{C^{(i)}}, C^{(1)}C^{(2)}\cdots C^{(k)}) < \epsilon$*, where $C^{(i)}$ are blocks of length $n$ ($i = 1,2,\ldots,k$).* ∎

ODOMETERS. Let $\overline{p} := (p_t)_{t\in\mathbb{N}}$ be a fixed increasing sequence of positive integers satisfying $p_t|p_{t+1}$. One of the possible ways of viewing the group $G_{\overline{p}}$ of $\overline{p}$-adic integers is the following: its elements are sequences $(j_t)_{t\in\mathbb{N}} \in \prod_{t\in\mathbb{N}}\{0,1\ldots,p_t-1\}$ such that for each $t$, $j_{t+1} \equiv j_t \bmod p_t$. Addition is defined coordinatewise (modulo $p_t$ on the $t$th axis). For $g \in G_{\overline{p}}$ and a natural $k$ we will write $kg$ instead of $g+g+\cdots+g$ ($k$ times). For each $t$ the set $p_tG_{\overline{p}}$ is a subgroup of $G_{\overline{p}}$ and $G_{\overline{p}}$ appears as the disjoint union of the sets $p_tG_{\overline{p}}+i$ ($i=0,1,\ldots,p_t-1$). The above sets are both open and closed, and, with $t$ ranging over all natural numbers, they form a base for the topology in $G_{\overline{p}}$. (There is another representation of $G_{\overline{p}}$, namely as a subset of $\{0,1,\ldots,p_1-1\}\times\prod_{t\in\mathbb{N}}\{0,1\ldots,\frac{p_{t+1}}{p_t}-1\}$ where addition involves carry to the right; we will not be using this notation; see, e.g., [H-R] for more details.)

The neutral element $(0,0,0,\ldots)$ of $G_{\overline{p}}$ will be denoted by 0. Similarly, 1 will be used to denote the element $(1,1,1,\ldots)$. We define a map $\tau\colon G_{\overline{p}} \mapsto G_{\overline{p}}$ by $\tau(g) := g+1$. The set $\mathbf{Z} := \{\tau^n(0) : n \in \mathbb{Z}\}$ (the orbit of 0) is a dense subgroup homomorphic to the additive group $\mathbb{Z}$ of the integers. We shall identify its elements with the integers $n \in \mathbb{Z}$. By an **odometer** (also called **adding machine**) we mean the topological dynamical system $(G_{\overline{p}},\tau)$. The system $(G_{\overline{p}},\tau)$ is strictly ergodic and the only invariant measure is the Haar measure $\lambda$.

Topologically, the group $G_{\overline{p}}$ can be identified with a Cantor set $\mathfrak{C}$ embedded in the unit interval in a classical way. We briefly sketch the construction: At first we divide the interval $[0,1]$ into $2p_1-1$ subintervals of equal lengths and we discard every second (open) subinterval. We are left with $p_1$ closed subintervals. Then, each of them we divide into $2\frac{p_2}{p_1}-1$ subintervals and we discard every second (open) subinterval. We are now left with $p_2$ closed intervals. And so on. The intersections of $\mathfrak{C}$ with the $p_t$ closed intervals obtained in step $t$ represent the partition of $G_{\overline{p}}$ into the cosets $p_tG_{\overline{p}}+i$ ($i=0,1,\ldots,p_t-1$) of the subgroup $p_tG_{\overline{p}}$.

The above embedding induces the order in $G_{\overline{p}}$ inherited from $[0,1]$. Further on, for $g_1,g_2 \in G_{\overline{p}}$, by $[g_1,g_2]$, $(g_1,g_2]$, etc., we shall understand the appropriate order intervals in $G_{\overline{p}}$. In this embedding the neutral element 0 is sent to $0 \in \mathfrak{C}$. The right endpoint of the Cantor set is the image of the element $-1$ in $\mathbf{Z}$. Generally, the elements of $\mathbf{Z}$ correspond to the endpoints of the constructed intervals.

The Haar measure $\lambda$ appears as a measure (also denoted by $\lambda$) on the interval, supported by $\mathfrak{C}$. The distribution function $F_\lambda$ of this measure restricted to $\mathfrak{C}$ has the following properties: it is continuous, non-decreasing, and the preimage of $\alpha \in [0,1]$ is either a single point not belonging to $\mathbf{Z}$ or a pair of points from

**Z**. It is easy to see that the second case happens precisely for numbers of the form $\alpha = kp_t/p_{t+s}$ ($k, t, s \in \mathbb{N}, 0 \le k \le p_{t+s}/p_t$). We call such numbers **$\overline{p}$-adic rationals**. In any case, by $g(\alpha)$ we shall denote the minimum of the preimage $F_\lambda^{-1}(\alpha)$, viewed as an element of $G_{\overline{p}}$.

*Definition 1:* A **fundamental interval** is an interval of the form either $[0, g(\alpha)]$ or $(g(\alpha), g(\beta)]$ $(0 < \alpha < \beta)$.

It is elementary to verify that an order interval $[0, b]$ or $(a, b]$ is fundamental if and only if its characteristic function is continuous at all points of **Z**. The sets $p_t G_{\overline{p}} + i$ are fundamental intervals.

TOEPLITZ SEQUENCES. We shall now recall some basic facts concerning Toeplitz sequences and Toeplitz flows, but only those which find direct application in this note. We refer the reader to [W] for a more complete exposition.

*Definition 2* ([J-K])*:* Let $\Sigma$ be a finite set. A **Toeplitz sequence** is an element $\omega \in \Sigma^{\mathbb{Z}}$ such that

$$(\forall n \in \mathbb{Z})(\exists p \in \mathbb{N})(\forall k \in \mathbb{Z})\ \omega(kp + n) = \omega(n),$$

i.e., each position in $\omega$ is a periodic position.

(It is customary to exclude periodic sequences, but for our purposes it will be convenient to view them as a special case of Toeplitz sequences.) A subshift $(X, \sigma)$ is called a **Toeplitz flow** if it is the shift orbit-closure of some Toeplitz sequence. Toeplitz flows are well-known to be minimal. Note that not all elements of a Toeplitz flow are Toeplitz sequences.

Non-periodic Toeplitz flows are characterized as totally disconnected expansive minimal almost 1-1 extensions over odometers (see [Mar] or [D-K-L]). There is another characterization of Toeplitz sequences (including periodic sequences) which exhibits their connection with odometers. We refer the reader to [D-I] for the proof.

FACT 3 ([D-I], Proposition 5): *A sequence $\omega \in \Sigma^{\mathbb{Z}}$ is Toeplitz if and only if there exists an odometer $G_{\overline{p}}$ and a Borel-measurable function $\eta\colon G_{\overline{p}} \mapsto \Sigma$ such that*

(1) *$\eta$ is continuous at each element of* **Z**,
(2) *$\omega = \eta|_{\mathbf{Z}}$ (i.e., $\omega(n) = \eta(n)$ for each $n \in \mathbb{Z}$).* ∎

It is not hard to see that $\omega$ is periodic with a period $p_t$ if and only if it is obtained as $\eta|_{\mathbf{Z}}$ for a continuous function on $G_{\overline{p}}$ (continuous functions are exactly those which are constant on the intervals $p_tG_{\overline{p}} + i$, $i = 0, 1, \ldots, p_t - 1$ for some index $t$).

At this point we choose an odometer $(G_{\overline{p}}, \tau)$ which will remain fixed for the rest of the paper.

Any Borel-measurable function $\eta$: $G_{\overline{p}} \mapsto \Sigma$, satisfying the condition (1) above, will be simply called a **Toeplitz function**. Note that the discontinuity points of any finite-valued function form a closed set, hence Toeplitz functions are continuous on a topologically large (dense open) set.

From now on we will consider exclusively Toeplitz sequences in the form $\eta|_{\mathbf{Z}}$. We will assign the attributes of the Toeplitz flow $(X_\eta, \sigma)$ generated by the Toeplitz sequence $\eta|_{\mathbf{Z}}$ directly to the Toeplitz function $\eta$. We shall thus say that $\eta$ is "strictly ergodic" or "has entropy zero", etc., hoping that this inexactitude will cause no confusion. If $\eta$ is strictly ergodic, then the corresponding invariant measure (on $\Sigma^{\mathbb{Z}}$) will be denoted by $\eta^*$. For instance, it is well-known that if the set of discontinuity points of $\eta$ has Haar measure zero, then $\eta$ is strictly ergodic and has entropy zero ($\eta|_{\mathbf{Z}}$ is then called a **regular Toeplitz sequence**, see [Mar]).

We remark that $(G_{\overline{p}}, \tau)$ is not always a topological factor of the Toeplitz flow $(X_\eta, \sigma)$. A sufficient condition is that the Toeplitz function $\eta$ is essentially NOT invariant under any rotation of $G_{\overline{p}}$, i.e., that the following holds:

(A) $\quad (\forall g' \in G_{\overline{p}} \setminus \{0\})\ \eta(g) \neq \eta(g + g')$ on an open set of elements $g \in G_{\overline{p}}$

(see [D2] or [D-D]). We will soon provide a technical device (the core) to ensure that all Toeplitz functions appearing in our constructions satisfy this condition. Moreover, this tool also secures that the map $\eta|_{\mathbf{Z}} \mapsto \eta^*$ becomes injective (in an appropriate class).

If $\eta$ is any Toeplitz function on $G_{\overline{p}}$ then, for each $t$, the Toeplitz sequence $\omega := \eta|_{\mathbf{Z}}$ splits naturally into blocks of the form $\omega[mp_t, (m+1)p_t - 1]$. We will call them **$t$-blocks** of $\eta|_{\mathbf{Z}}$. By a **$t$-code** we shall mean a function $\pi$ sending all blocks over $\Sigma$ of length $p_t$ into blocks of the same length over another alphabet. A $t$-code applies to a function $\eta$ on $G_{\overline{p}}$ as follows: For $g \in p_tG_{\overline{p}}$ consider the block $B := [\eta(g), \eta(g+1), \ldots, \eta(g + p_t - 1)]$. We set

$$\pi\eta(g + i) := \pi(B)(i),\ i = 0, 1, \ldots, p_t - 1.$$

Since $G_{\overline{p}}$ splits as the disjoint union of sets $p_tG_{\overline{p}}+i$, the above defines the function $\pi\eta$ on the entire group $G_{\overline{p}}$. It is easy to see that the image of a Toeplitz function is again a Toeplitz function. If $\eta$ is such that $(G_{\overline{p}}, \tau)$ is a topological factor of $(X_\eta, \sigma)$ (for example, if $\eta$ satisfies (A)) and $\pi$ is a $t$-code, then the Toeplitz flow $(X_{\pi\eta}, \sigma)$ is a topological factor of $(X_\eta, \sigma)$ (via the same code $\pi$ applied to $t$-blocks of the elements of $X_\eta$; we need $(G_{\overline{p}}, \tau)$ to be a factor of $(X_\eta, \sigma)$ in order to determine how each element of $X_\eta$ splits into $t$-blocks).

Whenever possible, we will avoid specifying the alphabet $\Sigma$, saying only that $\eta$ is a *finite-valued* function, or that $\omega$ is a sequence *over a finite alphabet.* This will allow us to freely add new symbols to the alphabet, create product alphabets, etc., without worrying about denoting the new alphabet. Most of the time we will consider finite vectors of Toeplitz functions $\eta = (\eta_1, \eta_2, \dots, \eta_d)$. Such a vector is, of course, also a Toeplitz function (into an appropriate product alphabet), and the corresponding Toeplitz sequence $\eta|_{\mathbf{Z}}$ splits into component Toeplitz sequences $\eta_i|_{\mathbf{Z}}$ which we call **rows**. We reserve the subscript indices to enumerate the rows. For instance, if $B$ is a block of $\eta$ then $B_i$ denotes its $i$th row which is a corresponding block of $\eta_i$. Concatenations of blocks will be thus indexed by superscripts: $B' = B^{(1)}B^{(2)}\cdots B^{(k)}$. Consequently, $B_i^{(j)}$ denotes the $i$th row of the $j$th component in a concatenation of multi-row blocks. (Brackets are used to avoid confusion with a customary notation of repetitions of a block, $B^k := \underbrace{BB\cdots B}_{k \text{ times}}$.)

QUASI-UNIFORM CONVERGENCE. The concept of quasi-uniform convergence was introduced by Jacobs and Keane in [J-K]. It was then exploited in [D-I]. Roughly speaking, two points in a dynamical system are considered close if their trajectories are close along a large lower Banach density subset of the time. In this work we will never use the original definition of the convergence, only its particular form valid for Toeplitz flows, namely the convergence in $\overline{R}$. Below we state the minimum of necessary facts concerning that notion (in an applicable version), and we refer the reader to [D-I] for the proofs. For a non-negative function $f$ on $G_{\overline{p}}$, by $\overline{R}(f)$ we denote the upper Riemann integral of $f$ with respect to the Haar measure $\lambda$. Note that if $f$ is a 0-1-valued function then

$$\overline{R}(f) = \lambda(\overline{\{g \in G_{\overline{p}} : f(g) = 1\}}).$$

By convention, the distance between elements in finite sets is 0 or 1. In particular, if $\eta, \eta'$ are finite-valued functions on $G_{\overline{p}}$ then $\overline{R}(\mathrm{dist}(\eta, \eta'))$ is the measure $\lambda$ of the closure of the set where $\eta$ differs from $\eta'$.

FACT 4 ([D-I], Lemma 5): *If $\eta_n$ and $\eta$ are Toeplitz functions on $G_{\overline{p}}$ and*

$$\overline{R}(\mathrm{dist}(\eta_n, \eta)) \to 0,$$

*then* $\eta_n|_{\mathbf{Z}} \to \eta|_{\mathbf{Z}}$ *quasi-uniformly.* ∎

Below, $h_{\rm top}(\omega)$ denotes the topological entropy of the shift transformation restricted to the orbit closure of $\omega$, while $K(\omega)$ denotes the simplex of invariant measures of that system.

FACT 5 ([D-I], Proposition 3 and Theorem 2): *Suppose that $\omega_n$ and $\omega$ are sequences over a (common) finite alphabet, and that $\omega_n \to \omega$ quasi-uniformly. Then $h_{\rm top}(\omega_n) \to h_{\rm top}(\omega)$, and $K(\omega_n) \to K(\omega)$ in the Hausdorff topology (over the weak* topology).* ∎

COROLLARY 1: *If, in Fact 4, the functions $\eta_n$ generate strictly ergodic Toeplitz flows, then so does $\eta$. Moreover, $\eta_n^* \to \eta^*$ and $h(\eta_n^*) \to h(\eta^*)$.* ∎

We will also need an observation concerning the $\overline{R}$-continuity of codes.

LEMMA 1: *If $\pi$ is a t-code and $(\eta_n)$ is a sequence of Toeplitz functions converging in $\overline{R}$ to a Toeplitz function $\eta$, then the images $\pi\eta_n$ converge in $\overline{R}$ to $\pi\eta$.*

*Proof:* It is easy to see that if $\overline{R}(\mathrm{dist}(\eta_n, \eta))$ is small, then the measure of the closure of the set

$$\{g \in p_t G_{\overline{p}} : (\eta_n(g), \eta_n(g+1), \ldots, \eta_n(g+p_t-1)) \neq (\eta(g), \eta(g+1), \ldots, \eta(g+p_t-1))\}$$

is also small, which yields that the measure of the closure of the set $\{g \in G_{\overline{p}} : \pi\eta_n(g) \neq \pi\eta(g)\}$ is small, too. ∎

## Technical tools

THE CORE. Our next notion is a simplified version of a concept introduced in [D1]. Simplification is possible, because in this work we do not care about maintaining the binary alphabet. Let $g_\diamondsuit$ be an element of $G_{\overline{p}} \setminus \mathbf{Z}$ which will remain fixed for the rest of this paper. Note that the interval $[0, g_\diamondsuit]$ is fundamental.

*Definition 3:* The 0-1-valued Toeplitz function $\eta_\diamondsuit$ equal to the characteristic function of the interval $[0, g_\diamondsuit]$ will be called the **core**. We say that a Toeplitz function $\eta$ (and the Toeplitz sequence $\eta|_{\mathbf{Z}}$) **has the core** if $\eta$ is a vector function and its first row is $\eta_\diamondsuit$.

The Toeplitz flow generated by $\eta_\diamondsuit$ will be denoted by $(X_\diamondsuit, \sigma)$. Each Toeplitz function $\eta$ can be "equipped" with the core simply by taking the vector Toeplitz function $(\eta_\diamondsuit, \eta)$.

LEMMA 2:

(1) *If $\eta$ has the core, then $(G_{\overline{p}}, \tau)$ is a topological factor of the Toeplitz flow $(X_\eta, \sigma)$ generated by $\eta$.*

(2) (Theorem 1 in [D1]) *Let $\eta$ and $\eta'$ be two Toeplitz functions with core. Then they generate the same Toeplitz flow $(X, \sigma)$ if and only if the induced Toeplitz sequences $\eta|_{\mathbf{Z}}$ and $\eta'|_{\mathbf{Z}}$ are equal.*

*Proof:* For (1) note that the Toeplitz flow $(X_\diamondsuit, \sigma)$ is a topological factor of $(X_\eta, \sigma)$ (by projection onto the first row). Thus it suffices to prove that $(X_\diamondsuit, \sigma)$ satisfies (A). To this end note that $g_\diamondsuit$ is the unique point where the limit of the function $\eta_\diamondsuit$ restricted to any dense subset does not exist, while $g_\diamondsuit - g'$ is the unique such point for the rotated function. Equality of these two functions on a dense set implies that $g' = 0$.

To prove (2), assume that $\eta'|_{\mathbf{Z}} = \lim \sigma^{n_k} \eta|_{\mathbf{Z}}$. Passing to a subsequence if necessary we can assume that $n_k$ converge in $G_{\overline{p}}$ to some $g'$. Then $\eta'(n) = \lim \eta(n + n_k) = \eta(n + g')$ whenever $n + g'$ is a continuity point of $\eta$. Because each $n \in \mathbf{Z}$ is a continuity point of $\eta$, it suffices to show that $g' = 0$. Repeating the above argument for the first row of $\eta$ (and of $\eta'$), i.e., to $\eta_\diamondsuit$, we obtain that $\eta_\diamondsuit(n) = \eta_\diamondsuit(n+g')$ except for at most one integer $n$ ($\eta_\diamondsuit$ has only one discontinuity point), i.e., on a dense set. But $\eta_\diamondsuit$ was just shown to satisfy (A), thus $g' = 0$. ∎

Recall that the map $\eta \mapsto \eta^*$ is defined on strictly ergodic Toeplitz functions and assigns to each of them the (unique) invariant measure carried by the generated Toeplitz flow. Clearly, $\eta^*$ depends exclusively on the Toeplitz sequence $\eta|_{\mathbf{Z}}$. From Lemma 2 we immediately obtain that:

COROLLARY 2: *The map $\eta|_{\mathbf{Z}} \mapsto \eta^*$ is 1-1 on strictly ergodic Toeplitz sequences with core.* ∎

$\overline{R}$-FACTORS. In [D1], whenever a strictly ergodic Toeplitz flow was needed, a regular one was used. Since all regular Toeplitz flows have entropy zero, we are forced to find a new tool. Namely, we shall fix one strictly ergodic Toeplitz flow of large entropy and further use only Toeplitz flows which are (close to) its factors. The details follow:

*Definition 4:* Let $\eta$ be a Toeplitz function. By an **$\overline{R}$-factor** of $\eta$ we will mean any other Toeplitz function $\eta'$ obtained as the $\overline{R}$-limit of images of $\eta$ by a sequence of $t$-codes (perhaps with varying indices $t$).

In particular, every continuous function or a Toeplitz function with countably many discontinuity points is an $\overline{R}$-factor of any Toeplitz function. It should be noted that an $\overline{R}$-limit of Toeplitz functions need not be a Toeplitz function, so this must be assumed separately. It follows easily from $\overline{R}$-continuity of the codes that being an $\overline{R}$-factor is a transitive relation.

In fact, we afford a slight abuse of terminology because, as mentioned above, a $t$-code is not always a "code" in the traditional meaning and need not lead to a topological factor. It does whenever $\eta$ is such that $(G_{\overline{p}}, \tau)$ is a topological factor of $(X_\eta, \sigma)$ (for example, if $\eta$ has the core, a case most common in the forthcoming considerations). To avoid introducing yet more terminology we use the term "$\overline{R}$-factor" even when the presence of the core is not assumed. Since a topological factor of a strictly ergodic flow is strictly ergodic, Corollary 1 implies that

LEMMA 3: *An $\overline{R}$-factor of a strictly ergodic Toeplitz function with core is strictly ergodic.* ∎

MIXTURE OF TOEPLITZ SEQUENCES. Our next tool is a kind of "convex combinations" of Toeplitz functions (hence also of Toeplitz sequences) which we call **mixtures**. This concept was introduced in [D-I] and then exploited in several papers of the first author. Part of Lemma 5 below is contained in [D1], Lemma 3(i), but the estimates for entropy are new. We present full definition and proofs. Let $\overline{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_d)$ be a probability vector, i.e., a finite sequence of non-negative numbers whose sum is 1. We divide $G_{\overline{p}}$ into $d$ subsets whose measures are $\alpha_1, \alpha_2, \ldots, \alpha_d$, respectively, in the following way: Fix some $t \in \mathbb{N}$. For $i = 1, 2, \ldots, d$ let

$$g_i := g\Big(\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_i}{p_t}\Big)$$

(recall that $g(\alpha)$ has been defined as a point for which $[0, g(\alpha)]$ has Haar measure $\alpha$). The Haar measure assigns to the fundamental intervals $[0, g_1], (g_1, g_2], \ldots, (g_{d-1}, g_d]$ in $G_{\overline{p}}$ the weights $\alpha_1/p_t, \alpha_2/p_t, \ldots, \alpha_d/p_t$ and those intervals form a partition of the subgroup $p_t G_{\overline{p}}$. We define

$$A_t(\overline{\alpha}, 1) := [0, g_1] \cup ([0, g_1] + 1) \cup ([0, g_1] + 2) \cup \cdots \cup ([0, g_1] + p_t - 1),$$

and, for $1 < i \leq d$,

$$A_t(\overline{\alpha}, i) := (g_{i-1}, g_i] \cup ((g_{i-1}, g_i] + 1) \cup ((g_{i-1}, g_i] + 2) \cup \cdots \cup ((g_{i-1}, g_i] + p_t - 1).$$

Below we sketch the set $\mathfrak{C}$ for $p_1 = 4$, $p_2 = 8$, $p_3 = 24$, $p_4 = 72$. For $\overline{\alpha} = (\frac{1}{9}, \frac{3}{9}, \frac{3}{9}, \frac{2}{9})$ the set $A_2(\overline{\alpha}, 3)$ is marked by underlining.



By construction, for any natural $m$ the integers $mp_t, mp_t + 1, \ldots, mp_t + p_t - 1$ belong to the same set $A_t(\overline{\alpha}, i)$. For each $i$, the integers contained in $A_t(\overline{\alpha}, i)$ have density $\alpha_i$ in $\mathbb{Z}$ (this follows from strict ergodicity of the system $(G_{\overline{p}}, \tau)$).

*Definition 5:* Let $\eta_1, \eta_2, \ldots, \eta_d$ be a finite family of Toeplitz functions defined on $G_{\overline{p}}$, and let $\overline{\alpha} \in \mathbb{R}^d$ be a probability vector. The **mixture** $\mathrm{Mix}_t(\overline{\alpha}, \eta_1, \eta_2, \ldots, \eta_d)$ is the function $\eta$ defined on $G_{\overline{p}}$ by

$$\eta(g) := \eta_i(g), \quad \text{if } g \in A_t(\overline{\alpha}, i) \quad (i = 1, 2, \ldots, d).$$

We remark that the Toeplitz functions $\eta_i$ used to produce a mixture will often be themselves vector functions; in particular, they will have the core. It is obvious that if each $\eta_i$ has the core then so does the mixture $\eta$. By the previously observed continuity property of characteristic functions of fundamental intervals, $\eta$ (regardless of $\overline{\alpha}$) is seen to be a Toeplitz function. The Toeplitz sequence $\eta|_{\mathbf{Z}}$ has the following properties: it is built by rewriting (without changing position) the $t$-blocks from the Toeplitz sequences $\eta_i|_{\mathbf{Z}}$; for each $i$ the $t$-blocks extracted from $\eta_i|_{\mathbf{Z}}$ occupy a subset of $\mathbb{Z}$ of density $\alpha_i$. Clearly, the map $\overline{\alpha} \mapsto \mathrm{Mix}_t(\overline{\alpha}, \eta_1, \eta_2, \ldots, \eta_d)$ is continuous with respect to $\overline{R}$. Thus, by Corollary 1, topological entropy and the set of invariant measures of the generated Toeplitz flow are continuous functions of the vector $\overline{\alpha}$. We shall soon prove that (under certain assumptions) for large $t$ these functions are also affine up to a preset accuracy. But first we need another observation:

LEMMA 4: *A mixture $\eta$ of $\overline{R}$-factors of a fixed Toeplitz function $\eta_\infty$ is an $\overline{R}$-factor of $\eta_\infty$.*

*Proof:* Suppose that each $\eta_i$ is an $\overline{R}$-factor of some fixed Toeplitz function $\eta_\infty$. Note that the vector-valued function $\overline{\eta} := (\eta_1, \eta_2, \ldots, \eta_d)$ is itself a Toeplitz function. Moreover, it is easy to see that $\overline{\eta}$ is an $\overline{R}$-factor of $\eta_\infty$ (a vector of images by codes is an image by a code, and $\overline{R}$ convergence for vectors is "coordinatewise"). Now suppose that all the coefficients $\alpha_i$ are $\overline{p}$-adic rationals. Then there exists an index $t' > t$ (where $t$ is the "mixing index" in $\mathrm{Mix}_t$) such that all the sets $A_t(\overline{\alpha}, i)$ are unions of sets of the form $p_{t'} G_{\overline{p}} + j$ $(0 \le j \le p_{t'} - 1)$. Thus there exists a $t'$-code $\pi$ sending $\overline{\eta}$ to $\eta$, namely each block of $\overline{\eta}$ is a $d \times p_t$

matrix; $\pi$ picks from the column $j$ a symbol from the row $i$ where $i$ satisfies the inclusion $p_{t'}G_{\overline{p}} + j \subset A_t(\overline{\alpha}, i)$.

Any other mixture is an $\overline{R}$-limit of the mixtures with $\overline{p}$-adic rational coefficients. ∎

LEMMA 5: *Consider a finite family of Toeplitz functions $\eta_1, \eta_2, \ldots, \eta_d$ where each $\eta_i$ is an $\overline{R}$-factor of a fixed strictly ergodic Toeplitz function $\eta_\infty$ with core. Then, for every $\epsilon > 0$, there exists an index $t$ such that for any probability vector $\overline{\alpha} \in \mathbb{R}^d$ the mixture $\eta := \mathrm{Mix}_t(\overline{\alpha}, \eta_1, \eta_2, \ldots, \eta_d)$ satisfies*

(1) $\mathrm{dist}\left(\eta^*, \sum_{i=1}^d \alpha_i \eta_i^*\right) < \epsilon$

*and*

(2) $\left|h(\eta^*) - h\left(\sum_{i=1}^d \alpha_i \eta_i^*\right)\right| < \epsilon$.

*The maps $\overline{\alpha} \mapsto \eta^*$ and $\overline{\alpha} \mapsto h(\eta^*)$ are continuous.*

*Proof:* By Lemma 4, $\eta$ is an $\overline{R}$-factor of $\eta_\infty$, hence, by Lemma 3, it is strictly ergodic so that the meaning of $\eta^*$ is definite. The last assertion follows directly from Corollary 1.

The proof of (1) is a direct application of Fact 2:

Let $t$ be such that $p_t$ is larger than the lengths $n_0$ appearing in the assertions (1) and (3) of Fact 2 for $\epsilon/3$ and $(X_{\eta_i}, \sigma)$ for each $i$. We use this index $t$ to produce the Toeplitz function $\eta := \mathrm{Mix}_t(\overline{\alpha}, \eta_1, \eta_2, \ldots, \eta_d)$. Let $\omega := \eta|_{\mathbf{Z}}$ and let $B' := \omega[0, kp_t]$ be so long that it approximates $\eta^*$ up to $\epsilon/3$. Note that $B'$ is a concatenation $B^{(1)}B^{(2)}\cdots B^{(k)}$ of $t$-blocks of $\eta_i|_{\mathbf{Z}}$ $(i = 1, 2, \ldots, d)$, the contribution of each $\eta_i|_{\mathbf{Z}}$ is nearly $\alpha_i$, and consequently

$$\begin{aligned}
&\mathrm{dist}\left(\eta^*, \sum_{i=1}^d \alpha_i \eta_i^*\right) \\
&\quad \leq \mathrm{dist}(\eta^*, B') + \mathrm{dist}\left(B', \frac{1}{k}\sum_{j=1}^k \mu_{B^{(j)}}\right) + \mathrm{dist}\left(\frac{1}{k}\sum_{j=1}^k \mu_{B^{(j)}}, \sum_{i=1}^d \alpha_i \eta_i^*\right) \\
&\quad < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.
\end{aligned}$$

We shall now prove (2). Part (1) and upper semi-continuity of the entropy function in symbolic systems imply that

$$h(\eta^*) < h\left(\sum_{i=1}^d \alpha_i \eta_i^*\right) + \epsilon$$

if $t$ is sufficiently large. In the proof of the reversed inequality of (2) we shall use the conditional variational principle (see [D-S]). Let $\mathcal{B}_t(\eta_i)$ denote the collection of all $t$-blocks of $\eta_i|_{\mathbf{Z}}$ ($\mathcal{B}_t(\eta)$ and $\mathcal{B}_t(\overline{\eta})$ will have analogous meaning). By strict ergodicity, the variational principle, and a well-known formula for the topological entropy of a subshift, we have

$$h(\eta_i^*) = \lim_{t\to\infty} \frac{1}{p_t} \log \#\mathcal{B}_t(\eta_i) \quad (i = 1, 2, \ldots, d),$$

and

$$h(\eta^*) = \lim_{t\to\infty} \frac{1}{p_t} \log \#\mathcal{B}_t(\eta),$$

while the topological entropy of the Toeplitz flow generated by $\overline{\eta} := (\eta_1, \eta_2, \ldots, \eta_d)$ equals

$$h_{\text{top}}(\overline{\eta}) = \lim_{t\to\infty} \frac{1}{p_t} \log \#\mathcal{B}_t(\overline{\eta}).$$

Now suppose that $\overline{\alpha}$ consists of $\overline{p}$-adic rational numbers, $\alpha_i = k_i p_t / p_{t+s}$ ($s \in \mathbb{N}$). Then every $(t+s)$-block of $\eta|_{\mathbf{Z}}$ decomposes into $t$-blocks of $\eta_i|_{\mathbf{Z}}$, and the number of $t$-blocks coming from $\eta_i|_{\mathbf{Z}}$ is $k_i$. Consider a $t$-block $D$ of $\overline{\eta}|_{\mathbf{Z}}$. It can be regarded as a $d \times p_t$ matrix such that, for each $i$, the $i$th row is a $t$-block of $\eta_i|_{\mathbf{Z}}$, this row denoted by $D_i$. By a straightforward interpretation of the definition introduced in [D-S], the topological conditional entropy of the Toeplitz flow generated by $\overline{\eta}|_{\mathbf{Z}}$ with respect to the factor generated by $\eta_i|_{\mathbf{Z}}$ is equal to

$$\lim_{t\to\infty} \sup_{B\in\mathcal{B}_t(\eta_i)} \frac{1}{p_t} \log \#\{D \in \mathcal{B}_t(\overline{\eta}) : D_i = B\}.$$

On the other hand, by the mentioned conditional variational principle and strict ergodicity of $\eta_i$, the above topological conditional entropy is also equal to $h_{\text{top}}(\overline{\eta}) - h(\eta_i^*)$. We can assume that $t$ is an index for which the element of the above displayed sequence differs from its limit by less than $\epsilon$, for each $i$:

$$\sup_{B\in\mathcal{B}_t(\eta_i)} \frac{1}{p_t} \log \#\{D \in \mathcal{B}_t(\overline{\eta}) : D_i = B\} < h_{\text{top}}(\overline{\eta}) - h(\eta_i^*) + \epsilon.$$

A given $(t+s)$-block $D'$ of $\overline{\eta}|_{\mathbf{Z}}$ gives rise to a $(t+s)$-block $B'$ of $\eta|_{\mathbf{Z}}$ in the following way: $D'$ is a concatenation $D^{(1)}D^{(2)}\cdots D^{(q)}$ (where $q := p_{t+s}/p_t$) of $t$-blocks of $\overline{\eta}|_{\mathbf{Z}}$. From each $D^{(j)}$ we pick one row $B^{(j)} := D^{(j)}_{i(j)}$(recall that $t$ is the "mixing index" in $\text{Mix}_t$), where the indices $i(j)$ do not depend on $D'$ (they are determined by the probability vector $\overline{\alpha}$). Then the concatenation $B' := B^{(1)}B^{(2)}\cdots B^{(q)}$ is a $(t+s)$-block of $\eta|_{\mathbf{Z}}$. The number of all $(t+s)$-blocks $D'$ of $\overline{\eta}|_{\mathbf{Z}}$ is equal to the sum, over all $(t+s)$-blocks $B'$ of $\eta|_{\mathbf{Z}}$, of the numbers of those blocks $D'$ which yield $B'$. Each summand is not larger than the product, over all component $t$-blocks $B^{(j)}$ of $B'$, of the numbers of all possible matrices $D$ having $B^{(j)}$ in its $i(j)$th row:

$$\#\mathcal{B}_{t+s}(\overline{\eta}|\mathbf{z}) \leq \sum_{B\in\mathcal{B}_{t+s}(\eta|\mathbf{z})} \prod_{j=1}^{p_{t+s}/p_t} \#\{D \in \mathcal{B}_t(\overline{\eta}) : D_{i(j)} = B^{(j)}\}.$$

Applying the previous estimate and the fact that, regardless of $B'$, $i(j)$ assumes a value $i$ exactly $k_i$ times, we obtain

$$\#\mathcal{B}_{t+s}(\overline{\eta}|\mathbf{z}) \leq \#\mathcal{B}_{t+s}(\eta|\mathbf{z}) \prod_{i=1}^{d} e^{[h_{\text{top}}(\overline{\eta})-h(\eta_i^*)+\epsilon]p_t k_i}.$$

This leads to the following estimate:

$$\frac{1}{p_{t+s}} \log \#\mathcal{B}_{t+s}(\overline{\eta}|\mathbf{z}) \leq \frac{1}{p_{t+s}} \log \#\mathcal{B}_{t+s}(\eta|\mathbf{z}) + \sum_{i=1}^{d} \alpha_i (h_{\text{top}}(\overline{\eta}) - h(\eta_i^*) + \epsilon).$$

Passing with $s$ to infinity, canceling out $h_{\text{top}}(\overline{\eta})$ and rearranging the items we obtain the desired lower bound for $h(\eta^*)$:

$$h(\eta^*) \geq \sum_{i=1}^{d} \alpha_i h(\eta_i^*) - \epsilon.$$

For remaining values of the vector $\overline{\alpha}$ the assertion now follows from continuous dependence of entropy on this vector. ∎

Lowering the entropy. In this section we show how to find, for a given $\overline{R}$-factor $\eta$, a nearby $\overline{R}$-factor with preset entropy smaller than $h(\eta^*)$.

Lemma 6: *Let $\eta$ be an $\overline{R}$-factor of $\eta_\infty$. For each $\epsilon > 0$ and $0 \leq \kappa \leq h(\eta^*)$ there exists an $\overline{R}$-factor $\tilde{\eta}$ of $\eta_\infty$ such that $\tilde{\eta}^*$ is within $\epsilon$-distance from $\eta^*$ and $h(\tilde{\eta}^*) = \kappa$. Moreover, $\tilde{\eta}|_{\mathbf{Z}}$ is built of $t$-blocks of $\eta|_{\mathbf{Z}}$ which are within $\epsilon$-distance from $\eta^*$.*

*Proof:* Find $t$ such that $p_t$ is larger than $n_0$ in (1) and (3) in Fact 2 for $\epsilon/2$. Fix some $t$-block $C$ of $\eta|_{\mathbf{Z}}$. Let $\eta_C$ denote the continuous function on $G_{\overline{p}}$ such that $\eta_C^* = \mu_C$ (recall that $\mu_C$ is the measure carried by the periodic orbit of $\cdots CCC \cdots$). Clearly, $h(\mu_C) = 0$, hence, for some $\alpha$, the mixture $\tilde{\eta} := \text{Mix}_t((\alpha, 1-\alpha), \eta, \eta_C)$ has entropy equal to $\kappa$. By Lemma 4, $\tilde{\eta}$ is still an $\overline{R}$-factor of $\eta_\infty$. By (3) in Fact 2, and convexity of the metric "dist", any measure generated by an infinite concatenation of $t$-blocks of $\eta|_{\mathbf{Z}}$ is within $\epsilon$-distance from $\eta^*$, and so is $\tilde{\eta}^*$. ∎

**Homeomorphic realization**

The following statement provides us with yet another tool for the final construction.

LEMMA 7 (comp. Theorem 1 in [D1]): *Let $K$ be a compact subset of a metric space. For every $0 < \gamma < 1$, not a $\overline{p}$-adic rational, there exists a map $x \mapsto \theta_x$ defined on $K$, into the collection of strictly ergodic 0-1 Toeplitz functions such that:*

(1) *for each $x \in K$, the Toeplitz function $\theta_x$ has at most countably many discontinuity points (hence is strictly ergodic),*
(2) *each $\theta_x$ assumes the value 0 on the interval $(g(\gamma), -1]$,*
(3) *the map $x \mapsto \theta_x|_{\mathbf{Z}}$ is injective,*
(4) *the map $x \mapsto \theta_x$ is continuous in $\overline{R}$.*

*Proof:* Let $a_1, a_2, \ldots$ be a sequence of strictly positive functions which separate points of $K$ and such that $\sum_k a_k \equiv \gamma$. Clearly, the map $x \mapsto (a_k(x))_{k\in\mathbb{N}}$ is an injective embedding of $K$ into the Hilbert cube. For each $x \in K$ let $\theta_x$ be the function on $G_{\overline{p}}$ defined as follows:

$$\theta_x(g) := \begin{cases} 1 & \text{if } g \in [0, b_1] \\ 0 & \text{if } g \in (b_{2i-1}, b_{2i}], \quad i \in \mathbb{N} \\ 1 & \text{if } g \in (b_{2i}, b_{2i+1}], \quad i \in \mathbb{N} \\ 0 & \text{if } g \in (g(\gamma), -1] \end{cases}$$

where $b_k := g\left(\sum_{i=1}^{k} a_i(x)\right)$. The value of $\theta_x$ at $g(\gamma)$ is inessential. As is easy to observe, every $\theta_x$ is a Toeplitz function. All required properties of this assignment follow immediately. ∎

It is well-known that for every $\kappa \geq 0$ there exists a Toeplitz sequence $\omega$ generating a strictly ergodic Toeplitz flow of entropy equal to $\kappa$ and which is an almost 1-1 extension of $(G_{\overline{p}}, \tau)$. Let $\eta_\infty$ be a Toeplitz function on $G_{\overline{p}}$ whose restriction to $\mathbf{Z}$ is $\omega$. We assign to each $x \in K$ the vector Toeplitz function with three rows:

$$\eta_x := (\eta_\diamond, \theta_x, \eta_\infty).$$

Each $\eta_x$ (or $\eta_x|_{\mathbf{Z}}$) has three rows: the first row is the core, the second row with countably many discontinuities is responsible for the map $x \mapsto \eta_x$ being injective, and the third row is responsible for entropy. Below we summarize the properties of this new assignment.

COROLLARY 3: *With the assumptions of Lemma 7, and for any positive $\kappa$, there exists a map $x \mapsto \eta_x$ from $K$ into three-row (with core) $\overline{R}$-factors of some strictly ergodic Toeplitz function $\eta_\infty$, such that $x \mapsto \eta_x^*$ is a homeomorphism (onto its image). Moreover, the entropy of every measure $\eta_x^*$ is equal to $\kappa$.*

*Proof:* Since the function $\eta_\Diamond$ and each $\theta_x$ have countably many discontinuities, they are $\overline{R}$-factors of any Toeplitz function defined on $G_{\overline{p}}$. Thus $\eta_x$ is an $\overline{R}$-factor of $\eta_\infty$. Since every $\eta_x$ has the core, Corollary 2 and Lemma 7 imply that the map $x \mapsto \eta_x^*$ is injective. It is also continuous by Corollary 1. The statement concerning entropy is trivial (entropy of the first two rows is zero). ∎

### The expansive case

We are about to present a construction of a subshift with a preset pair $(K, h)$, where $h$ is upper semi-continuous. But first we shall need an elementary lemma concerning u.s.c. functions on compact sets:

LEMMA 8: *Let $f$ be a continuous real-valued function defined on a compact subset $K$ of a metric space $M$. Let $h$ be a u.s.c. function defined on $M$ such that $h < f$ on $K$. Then there exists a positive number $\xi$ such that*

$$(x \in K,\ y \in M,\ d(x,y) < \xi) \Longrightarrow h(y) < f(x).$$

*Proof:* If not, then we can find two sequences of points: $(x_n)$ in $K$ and $(y_n)$ in $M$, converging to a common limit $x_0$, such that $h(y_n) \geq f(x_n)$. Then using upper semi-continuity, the assumed inequality, and continuity, respectively, we obtain

$$h(x_0) = h(\lim y_n) \geq \overline{\lim} h(y_n) \geq \lim f(x_n) = f(x_0),$$

a contradiction, because $x_0$ belongs to $K$. ∎

The first main result of this work follows now:

THEOREM 1: *Let $K$ be a compact metrizable Choquet simplex, and let $f$ be a (bounded) non-negative affine u.s.c. function on $K$. Then there exists a Toeplitz flow $(Y, \sigma)$ (a minimal symbolic almost 1-1 extension of $(G_{\overline{p}}, \tau)$), and an affine (onto) homeomorphism $\phi^*\colon K \mapsto K(Y, \sigma)$, such that for every $x \in K$, $f(x) = h(\phi^*(x))$, where $h$ denotes the entropy function on $K(Y, \sigma)$.*

*Proof:* The proof will be divided into several labeled *stages*.

STAGE 1. PRELIMINARY RESHAPING: By the properties of $f$, we can find a strictly decreasing sequence $(f_n)_{n\geq 1}$ of continuous affine functions on $K$, converging to $f$. According to Theorem 9 in [E], we can represent $K$ as the intersection of a descending sequence of metrizable Bauer simplices $(B_n)_{n\geq 1}$ in some locally convex linear space. (Recall that a Bauer simplex is a Choquet simplex whose extreme points form a closed set.) By the Hahn–Banach theorem, each $f_n$ can be extended to an affine continuous function defined on the largest simplex $B_1$. Replacing, if necessary, $(B_n)$ by a subsequence, we can arrange that $0 < f_{n+1} < f_n$ on $B_{n+1}$, for every $n \geq 1$. Additionally, we let $f_0$ be a constant function (equal to some $\kappa$) defined on $B_1$, strictly larger than $f_1$. We choose a decreasing to zero sequence $(\rho_n)_{n\geq 0}$ of positive numbers such that

$$\text{(i)} \qquad f_n < f_{n-1} - \rho_{n-1} \quad \text{on } B_n$$

for any $n \geq 1$.

STAGE 2. INDUCTIVE CONSTRUCTION: We will construct inductively a sequence of homeomorphic maps $\phi_n$ defined on $\operatorname{ex} B_n$ and taking values in the set of three-row $\overline{R}$-factors of some strictly ergodic Toeplitz function $\eta_\infty$. Then we define the maps $\phi_n^*$ in a natural way, by $\phi_n^*(e) := \eta^*$, where $\eta := \phi_n(e)$ $(e \in \operatorname{ex} B_n)$. The affine extensions of $\phi_n^*$ will converge on $K$ to an affine homeomorphism. Each image $\phi_n(e)$ will have three rows: the first with the core, the second row responsible for the maps being injective, and the third row where we play with the entropy.

STEP 1: Fix some $\xi_0$, not a $\overline{p}$-adic rational. Corollary 3 applied to the compact set $\operatorname{ex} B_1$ and $\gamma = \xi_0$ yields a map $\phi_1$ from $\operatorname{ex} B_1$ into strictly ergodic (three-row) Toeplitz functions with core, of entropy equal to $\kappa$ (i.e., to the constant function $f_0$), $\overline{R}$-factors of a fixed strictly ergodic Toeplitz function $\eta_\infty$. Moreover, the map $\phi_1^*$ is a homeomorphism. Consider the affine extension of that map to all of $B_1$:

$$\phi_1^*(x) := \int_{\operatorname{ex} B_1} \phi_1^*(e)\,d\nu_x(e),$$

where $\nu_x$ is the unique measure on $\operatorname{ex} B_1$ with barycenter at $x$. Since in Bauer simplices (unlike in general Choquet simplices) the map $x \mapsto \nu_x$ is continuous, the extended map $\phi_1^*$ is also continuous on $B_1$. As the extreme points of $B_1$ are sent to ergodic measures (hence extreme points among all invariant measures), the map $\phi_1^*$ is an isomorphism (i.e., an affine homeomorphism) of Bauer simplices.

*Inductive assumption:* Suppose that for some $n \geq 1$ we have defined a map $\phi_n$ from $\operatorname{ex} B_n$ into (three-row) $\overline{R}$-factors of $\eta_\infty$ having the core in the first row, at most countably many discontinuities in the second row, so that $\phi_n^*$ is a homeomorphism, and

$$\text{(ii)} \qquad f_n(e) < h(\phi_n^*(e)) < f_{n-1}(e) + \rho_{n-1},$$

for every $e \in \operatorname{ex} B_n$.

As before, we extend $\phi_n^*$ to an affine homeomorphism on all of $B_n$. Clearly, the inequality (ii) remains valid for this extended map (all involved functions are affine). We also assume that a positive number $\xi_{n-1}$ has been chosen.

*Inductive step:* We shall soon construct a map $\phi_{n+1}$ with the properties like in the inductive assumption (with $n$ replaced by $n+1$). Moreover, we will assure that the new map differs from the old one by a controlled uniform distance. First we need to choose a few auxiliary positive numbers. Let $0 < \xi_n < \xi_{n-1}/3$ be so small that

$$\text{(iii)} \qquad d(x,y) \geq \rho_n \Longrightarrow \operatorname{dist}(\phi_n^*(x), \phi_n^*(y)) \geq \xi_n$$

for any $x, y \in B_n$, and

$$\text{(iv)} \qquad \operatorname{dist}(\phi_n^*(x), \mu) < \xi_n \Longrightarrow h(\mu) < f_{n-1}(x) + \rho_{n-1},$$

for any $x$ in $B_n$ and any shift-invariant measure $\mu$ (see (ii) and Lemma 8). Choose $0 < \epsilon < \min\{\rho_n/2, \xi_n/4\}$, not a $\overline{p}$-adic rational. Finally, pick a $\delta > 0$ such that if $x, y \in B_n$ then

$$\text{(v)} \qquad d(x,y) < \delta \Longrightarrow \operatorname{dist}(\phi_n^*(x), \phi_n^*(y)) < \epsilon \quad \text{and} \quad |f_n(x) - f_n(y)| < \epsilon.$$

We are in position to start the construction:
Choose a finite $\delta$-dense subset $\{e_1, e_2, \ldots, e_d\}$ in $\operatorname{ex} B_n$. Let $S$ denote the finite-dimensional simplex $S := \operatorname{conv}\{e_1, e_2, \ldots, e_d\}$. For every point $x \in B_n$ the set $S_x := \{y \in S : d(x,y) < \delta\}$ is nonempty, convex, and the multifunction $x \mapsto S_x$ is lower semi-continuous. Applying Theorem 3.1$'''$ in [Mic], we can choose a continuous selector $s$: $B_n \mapsto S$ such that $d(s(x), x) < \delta$. For each $x$, $s(x)$ can be written as a convex combination $\sum_{i=1}^d \alpha_i(x) e_i$, with a continuous coefficient vector

$$\overline{\alpha}(x) := (\alpha_1(x), \alpha_2(x), \ldots, \alpha_d(x)).$$

Let $\eta_i := \phi_n(e_i)$. By the inductive assumption (ii) we have $f_n(e_i) < h(\eta_i^*)$. Thus, using Lemma 6 we can replace each Toeplitz function $\eta_i$ by another $\overline{R}$-factor of $\eta_\infty$, say $\tilde{\eta}_i$, built of $t$-blocks of $\eta_i$ approximating the measure $\eta_i^*$ up to $\epsilon$, and such that $\tilde{\eta}_i^*$ has entropy equal to $f_n(e_i)$. The parameter $t$ in this statement can be chosen common for all $i$. To each probability $d$-vector $\overline{\alpha}$ we associate the point $s_{\overline{\alpha}} := \sum_{i=1}^d \alpha_i e_i \in S$ and define

$$\tilde{\phi}_n^*(s_{\overline{\alpha}}) := \sum_{i=1}^d \alpha_i \tilde{\eta}_i^*.$$

Clearly, since $h\left(\tilde{\phi}_n^*(e_i)\right) = h(\tilde{\eta}_i^*) = f_n(e_i)$, we have

$$\text{(vi)} \qquad h\left(\tilde{\phi}_n^*(s)\right) = f_n(s)$$

on $S$. Moreover, since we are using a convex metric on measures,

$$\text{(vi}'\text{)} \qquad \operatorname{dist}\left(\phi_n^*(s), \tilde{\phi}_n^*(s)\right) < \epsilon.$$

By a direct application of Lemma 5, for a suitable index $t' \geq t$, the map

$$\psi(s_{\overline{\alpha}}) := \operatorname{Mix}_{t'}(\overline{\alpha}, \tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_d)$$

sends the points of $S$ to $\overline{R}$-factors of $\eta_\infty$ so that

$$\text{(vii)} \qquad \operatorname{dist}\left(\tilde{\phi}_n^*(s), \psi^*(s)\right) < \epsilon,$$

and

$$\text{(vii}'\text{)} \qquad \left|h(\tilde{\phi}_n^*(s)) - h(\psi^*(s))\right| < \epsilon,$$

and the map $\psi^*$ is continuous. Moreover, all images of the map $\psi$ induce sequences built of the $t'$-blocks appearing in $\tilde{\eta}_i$ and hence of $t$-blocks of $\phi_n(e_i)$ approximating $\phi_n^*(e_i)$ up to $\epsilon$ $(i = 1, 2, \ldots, d)$. The composition

$$x \mapsto \psi(s(x))$$

is defined on all of $B_n$, hence also on $\operatorname{ex} B_{n+1}$. It has all required properties of $\phi_{n+1}$ (which we discuss in a moment) except being injective. Thus we need to slightly modify the middle row. Let $\theta'_x$ denote the middle row of $\psi(s(x))$, and let $\theta_x^\gamma$ be the homeomorphism of Lemma 7 applied to the compact set $\operatorname{ex} B_{n+1}$ and $\gamma$ (which we specify in a moment). For each $x \in \operatorname{ex} B_{n+1}$ we define $\theta_x$ as

$$\theta_x(g) := \begin{cases} \theta_x^\gamma(g) & \text{if } g \in [0, g(\gamma)], \\ \theta'_x(g) & \text{if } g \in (g(\gamma), -1]. \end{cases}$$

We let $\phi_{n+1}(x)$ be the three-row Toeplitz function obtained from $\psi(s(x))$ by replacing its middle row $\theta'_x$ by $\theta_x$. By another application of $\overline{R}$-continuity it is seen that if $\gamma$ is chosen small enough then

$$\text{(viii)} \qquad \operatorname{dist}\left(\psi(s(x)), \phi_{n+1}(x)\right) < \epsilon.$$

Clearly, the entropy of $\phi^*_{n+1}(x)$ is the same as that of $\psi^*(s(x))$, for every $x \in \operatorname{ex} B_{n+1}$. Moreover, if $\gamma < 1/p_t$ then $[0, g(\gamma)] \subset p_t G_{\overline{p}}$, hence

(ix) $\phi_{n+1}(x)|_{\mathbf{Z}}$ is built of the $t$-blocks of $\phi_n(e_i)|_{\mathbf{Z}}$ $(i = 1, 2, \dots, d)$ changed (perhaps) only at the first position (in the middle row).

Still, each $\phi_{n+1}(x)$ is an $\overline{R}$-factor of $\eta_\infty$, the map $\phi_{n+1}$ is injective (because so is $x \mapsto \theta^\gamma_x|_{[0,g(\gamma)]}$) and continuous (because so are all component assignments). It remains to verify the estimates. Fix an element $e$ of $\operatorname{ex} B_{n+1}$ and consider the following sequence of measures:

$$\phi^*_n(e) \longrightarrow \phi^*_n(s(e)) \longrightarrow \tilde{\phi}^*_n(s(e)) \longrightarrow \psi^*(s(e)) \longrightarrow \phi^*_{n+1}(e).$$

Each arrow represents a move by at most $\epsilon$. The first one — by (v) and because $d(e, s(e)) < \delta$, the second — by (vi′), third — by (vii), and fourth — by (viii). This means that

$$\text{(x)} \qquad \operatorname{dist}\left(\phi^*_n(e), \phi^*_{n+1}(e)\right) < 4\epsilon < \xi_n.$$

By convexity of the metric, the above also holds for the extended affine map $\phi^*_{n+1}$ on all of $B_{n+1}$.

For entropies, we start from the central term $\tilde{\phi}^*_n(s(e))$, whose entropy equals $f_n(s(e))$ (see (vi)), which is within $\epsilon$ from $f_n(e)$ (see (v)). By (vii′), $h(\psi^*(s(e)))$ differs from that by at most another $\epsilon$. Finally, we pass to $\phi^*_{n+1}(e)$ without change in entropy. Since $2\epsilon < \rho_n$, we have shown that

$$f_{n+1}(e) < f_n(e) - \rho_n < h(\phi^*_{n+1}(e)) < f_n(e) + \rho_n$$

(see (i) for the first inequality) and the inductive assumption (ii) is fulfilled for $n + 1$.

Stage 3. The limit map: Obviously, on the Choquet simplex $K$ we have defined an infinite sequence of maps $\phi^*_n$. Because the uniform distance between the $n$th and $(n+1)$st map is not larger than $\xi_n$, and since $\xi_n$ form a convergent series (recall that $\xi_n < \xi_{n-1}/3$), these maps converge on $K$ uniformly to a map

$\phi^*$ (with values in the set of the shift-invariant measures). Clearly, this map is continuous and affine. Moreover, the uniform distance between $\phi^*_n$ and $\phi^*_m$ satisfies, for $n < m$,

$$\text{(xi)} \qquad \operatorname{dist}(\phi^*_n, \phi^*_m) \leq \sum_{k=n+1}^{\infty} \xi_k < \xi_n/2.$$

Obviously, the same holds, if $\phi^*_m$ is replaced by $\phi^*$. It follows that $\phi^*$ is also 1-1. Indeed, if $x \neq y$ then $d(x,y) \geq \rho_n$ for some $n$. By (iii), the $n$th map separates the images by $\xi_n$. The limit map can move each of them by less than $\xi_n/2$, thus they remain distinct. We have proved that $\phi^*$ is an isomorphism of Choquet simplices.

For entropies, the first inequality in (ii) and upper semi-continuity of the entropy function yield $f(x) \leq h(\phi^*(x))$. On the other hand, (xi) and (iv) imply that $h(\phi^*(x)) < f_{n-1}(x) + \rho_{n-1}$ for every $n$, hence $h(\phi^*(x)) = f(x)$.

STAGE 4. THE DYNAMICAL SYSTEM $(Y, \sigma)$: We will now indicate a symbolic dynamical system $(Y, \sigma)$ whose set of invariant measures coincides with the image $\phi^*(K)$. Moreover, we will show that $(Y, \sigma)$ is a topological extension of $(G_{\overline{p}}, \tau)$. The idea is very natural: we let

$$Y := \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} \bigcup_{e\in \operatorname{ex} B_k} O(\phi_k(e)|_{\mathbf{Z}})},$$

where $O(\omega)$ denotes the shift-orbit of a sequence $\omega$. The fact that for every $x \in K$ the measure $\phi^*(x)$ is supported by $Y$ follows immediately from the well-known lower semi-continuity of the map assigning to a measure its topological support; the topological support of $\phi^*_k(x)$ is contained in $\overline{\bigcup_{e\in \operatorname{ex} B_k} O(\phi_k(e)|_{\mathbf{Z}})}$.

We need to prove the converse, i.e., that every invariant measure on $Y$ has the form $\phi^*(x)$ ($x \in K$). This inclusion does not follow from any general fact (it may happen that a limit of invariant sets supports an invariant measure not approximated by measures supported by these sets), and we need to refer to some particularities of our construction. First observe that it suffices to examine only the ergodic measures carried by $Y$. For an ergodic measure $\nu$ we find in $Y$ a long block $C$ well approximating this measure. We can arrange that the length of $C$ is $kp_t$, where $t$ is the index appearing in a far inductive step $n+1$. By definition of $Y$, $C$ occurs in $\phi_{n'}(e)|_{\mathbf{Z}}$ for some $n' > n$ and $e \in \operatorname{ex} B_{n'}$. But, by a recursive application of (ix), everything built after step $n$ is a concatenation of the $t$-blocks appearing in $\phi_n(e_i)$ ($i = 1, 2, \ldots, d$), changed perhaps at the first position (this inaccuracy we can ignore). Thus, by (3) in Fact 2, and convexity of the metric, $C$ (hence also $\nu$) is within a small distance from a convex combination of measures appearing in $\phi_n^*(\operatorname{ex} B_n)$, hence from a measure $\phi_n^*(x_n)$ ($x_n \in B_n$). We have obtained that $\nu = \lim_n \phi_n^*(x_n)$. We can assume that $x_n$ converge to some $x \in K$. Then, for $m > n$ the map $\phi_n^*$ is defined at $x_m$ and we have the following approximation:

$$\nu \longrightarrow \phi_m^*(x_m) \longrightarrow \phi_n^*(x_m) \longrightarrow \phi_n^*(x) \longrightarrow \phi^*(x),$$

where each arrow represents a small move (use (xi)). Thus $\nu = \phi^*(x)$, as required.

What remains to show is that $(Y, \sigma)$ has $(G_{\overline{p}}, \tau)$ as a factor. This follows immediately, because $Y$ projects (by projection on the first row) onto the Toeplitz flow $(X_\Diamond, \sigma)$, which has $(G_{\overline{p}}, \tau)$ as a factor.

At the end we apply Fact 1 and we replace $(Y, \sigma)$ by a Borel* conjugate Toeplitz flow. This completes the proof. ∎

## The general case

Our approach to the general case is based on constructing multi-row subshifts which are $\overline{R}$-factors of a strictly ergodic dynamical system with infinite entropy. We shall need to modify some of our technical tools.

*Definition 6:* By a **vector Toeplitz function** we shall mean a countable (or finite) vector of Toeplitz functions $\overline{\eta} := (\eta_1, \eta_2, \ldots)$ (or $\eta := (\eta_1, \eta_2, \ldots, \eta_n)$; from now on the bar is reserved to indicate Toeplitz functions with infinitely many rows).

A vector Toeplitz function can be interpreted as a function into the compact metric space obtained as the product of the finite alphabets used by the functions $\eta_j$. By adjoining to each of these alphabets an additional symbol (interpreted as "nothing") we can view all vector functions as functions with infinitely many rows. For a function into such an alphabet to be a Toeplitz function is again equivalent to the condition of pointwise continuity at each element of $\mathbf{Z}$. We say that a sequence $\overline{\eta}_n$ of such vector functions **converges in** $\overline{R}$ to some vector function $\overline{\eta}$ if $\operatorname{dist}(\overline{\eta}_n, \overline{\eta}) \to 0$ in $\overline{R}$. In this sense vector functions with finitely many rows (but with the number of rows tending to infinity) may converge in $\overline{R}$ to a vector function with countably many (non-void) rows. The sequence $\overline{\eta}|_{\mathbf{Z}}$ can be interpreted as a sequence over a compact totally disconnected space. So, it generates (by shift-orbit closure) a totally disconnected dynamical system which is a topological joining of the Toeplitz flows generated individually by the rows $\eta_j|_{\mathbf{Z}}$.

For any finite subset $A$ of $\mathbb{N}$, $A = \{a_1, a_2, \dots, a_n\}$, the vector Toeplitz function $\eta_A := (\eta_{a_1}, \eta_{a_2}, \dots, \eta_{a_n})$ generates a dynamical system which is a topological factor of the one generated by $\overline{\eta}$. We can apply a $t$-code to a vector Toeplitz function $\overline{\eta}$ by applying it to $\eta_{\{1,2,\dots,n\}}$ for some finite $n$. By an **$\overline{R}$-factor** of $\overline{\eta}$ we shall understand any $\overline{R}$-limit of a sequence of so obtained images by $t$-codes ($t$ may vary).

A vector Toeplitz function is said to have the core if its first row is the core $\eta_\diamondsuit$. Clearly, if $\overline{\eta}$ has the core then $(G_{\overline{p}}, \tau)$ is a topological factor of the dynamical system generated by $\overline{\eta}|_{\mathbf{Z}}$. Adding the core (as the new first row) to a vector Toeplitz function $\overline{\eta}$ results in an $\overline{R}$-factor of $\overline{\eta}$.

Similarly as for Toeplitz function over a finite alphabet (Lemma 3), an $\overline{R}$-factor of a strictly ergodic vector Toeplitz function with core is strictly ergodic.

LEMMA 9: *There exists a strictly ergodic vector Toeplitz function $\overline{\eta}_\infty$ with core, whose entropy is infinite.*

*Proof:* By the Jewett–Krieger Theorem and Theorem 18 in [D-K], there exists a totally disconnected strictly ergodic realization of an infinite entropy weakly mixing system. Since such a system is disjoint from any odometer, its product with the odometer $(G_{\overline{p}}, \tau)$ remains strictly ergodic. By [D-L] we can find a minimal almost 1-1 extension $(X, T)$ of $(G_{\overline{p}}, \tau)$ which is Borel* conjugate to that product, hence strictly ergodic with infinite entropy. Moreover, the construction in [D-L] starting from a totally disconnected system produces a totally disconnected system. Now, $(X, T)$ can be represented as a subshift with countably many symbolic rows. Each row is a factor of $(X, T)$, hence by Theorem 9.13 and Proposition 9.9 in [F] (see also [D-D] for a direct proof) it is an almost 1-1 extension of a factor of $(G_{\overline{p}}, \tau)$, so the $j$th row can be represented as a system generated by a Toeplitz function $\eta_j$. Thus $(X, T)$ is generated by the vector Toeplitz function $\overline{\eta} := (\eta_1, \eta_2, \dots)$. Since $(G_{\overline{p}}, \tau)$ is a topological factor of $(X, T)$, the Toeplitz function $\overline{\eta}_\infty$ obtained by adding the core to $\overline{\eta}$ remains strictly ergodic. ∎

For a given $\overline{\eta}$ and two finite subsets $A_1, A_2$ of $\mathbb{N}$ (usually both of them of the form $\{n, n+1, \dots, n+k\}$) we will consider measure-theoretic conditional entropy of the form $h(\eta^*_{A_2}|\eta^*_{A_1}) := h(\eta^*_{A_1 \cup A_2}) - h(\eta^*_{A_1})$. We shall denote

$$h_1(\overline{\eta}^*) := h(\eta^*_1), \quad \text{and}$$
$$h_j(\overline{\eta}^*) := h(\eta^*_j|\eta^*_{\{1,2,\dots,j-1\}}) \quad \text{for } j \geq 2.$$

LEMMA 10 (Selective lowering of entropy): *Let $\overline{\eta} := (\eta_1, \eta_2, \dots)$ be a strictly ergodic vector Toeplitz function. Fix two integers $d$ and $m$, $1 \leq m \leq d$. Then,* *for every $\delta_h > 0$ and $\kappa \le h_m(\overline{\eta}^*)$ there exists an $\overline{R}$-factor $\overline{\eta}' = (\eta'_1, \eta'_2, \dots)$ of $\overline{\eta}$ within $\delta_h$ distance from $\overline{\eta}$ (we mean the distance of measures), such that*

$$\begin{aligned} &h_j(\overline{\eta}'^*) = h_j(\overline{\eta}^*) \quad \text{for } 1 \le j < m, \\ &h_m(\overline{\eta}'^*) = \kappa, \\ &|h_j(\overline{\eta}'^*) - h_j(\overline{\eta}^*)| < \delta_h \quad \text{for } m < j \le d. \end{aligned}$$

*Proof:* Denote $\eta := \overline{\eta}_{\{1,2,\dots,d\}}$. It suffices to prove the theorem for $\eta$ and find its appropriate $\overline{R}$-factor $\eta'$ (with $d$ rows). Then $\overline{\eta}'$ will be obtained by attaching to $\eta'$ the unchanged rows $\eta_{d+1}, \eta_{d+2}, \dots$ of $\overline{\eta}$.

By upper semi-continuity of the conditional entropy functions in expansive systems (see [D-S]), we can find $\delta$ so small that any vector Toeplitz function $\eta'$ within $\delta$ distance from $\eta$ has all conditional entropies $h_j(\eta'^*)$ ($j = 1, \dots, d$) smaller than the corresponding values for $\eta$ plus $\delta_h/2d$. Find an index $t$ such that (1) and (3) of Fact 2 hold for $\epsilon = \delta$ and $n_0 = p_t$.

The $t$-blocks of $\eta|_{\mathbf{Z}}$ can be written as triples $[B, C, D]$, with $B \in \mathcal{B}_t(\eta_{\{1,2,\dots,m-1\}})$, $C \in \mathcal{B}_t(\eta_m)$, and $D \in \mathcal{B}_t(\eta_{\{m+1,\dots,d\}})$ (if $m = 1$ or $m = d$ then $B$ or $D$ is the "empty" block, respectively). For each pair $[B, C]$ let $M([B, C])$ be the number of available (in such triples) blocks $D$. We can label these blocks $D$ (independently for each pair $[B, C]$) by natural numbers $1, 2, \dots, M([B, C])$. Now, every $t$-block $[B, C, D]$ can be encoded (injectively) as the triple $[B, C, l]$, where $l$ is the label of $D$ given the pair $[B, C]$. Thus the Toeplitz sequence $\eta|_{\mathbf{Z}}$ has a conjugate representation as the Toeplitz sequence with $m + 1$ rows $(\eta_{\{1,2,\dots,m\}}|_{\mathbf{Z}}, J)$, where $J$ is a one-row sequence built of some additional $t$-blocks assigned bijectively to the labels $l$, for example of the form $l, l, \dots, l$ ($p_t$ times). For each $B$ we pick one block $C^B$ such that $M([B, C^B])$ is maximal among all numbers $M([B, C])$.

Now we build a new sequence $\eta'|_{\mathbf{Z}}$ (using the $t$-blocks of $\eta|_{\mathbf{Z}}$) with "inessential" $m$th row and "unchanged" remaining rows: namely, in $\eta$ we replace each $t$-block $[B, C, l]$ by $[B, C^B, l]$ (since $[B, C^B]$ admits at least as many labels as $[B, C]$ does, $[B, C^B, l]$ represents some $t$-block of $\eta$). Such a procedure is obviously a $t$-code, hence $\eta'$ is an $\overline{R}$-factor of $\eta$.

Clearly, the conditional entropies involving the first $m - 1$ rows of $\eta'$ are the same as in $\eta$ (these rows are unchanged). Next, $h_m(\eta'^*) = 0$, because $C^B$ is determined by $B$. Note that the row $J$ (in appropriate representation) has not changed, either. We have

$$\begin{aligned} h(\eta'^*_{\{m+1,\dots,d\}}|\eta'^*_{\{1,2,\dots,m\}}) &= h(J^*|\eta'^*_{\{1,2,\dots,m\}}) = h(J^*|\eta^*_{\{1,2,\dots,m-1\}}) \\ &\ge h(J^*|\eta^*_{\{1,2,\dots,m\}}) = h(\eta^*_{\{m+1,\dots,d\}}|\eta^*_{\{1,2,\dots,m\}}). \end{aligned}$$

By Fact 2 and because $\eta'|_{\mathbf{Z}}$ is built of $t$-blocks of $\eta|_{\mathbf{Z}}$, its measure $\eta'^*$ is $\delta$-close to $\eta^*$, hence all conditional entropies $h_j(\eta'^*)$ are not larger than the corresponding entropies for $\eta$ plus $\delta_h/2d$. Since their sum over $j = m+1, \dots, d$ (equal to $h(\eta'^*_{\{m+1,\dots,d\}}|\eta'^*_{\{1,2,\dots,m\}})$) has not dropped from the analogous sum for $\eta$, none of the summands could drop by more than $\delta_h/2$. We have proved that, for $j = m+1, \dots, d$,

$$|h_j(\eta'^*) - h_j(\eta^*)| < \delta_h/2.$$

Now we use Lemma 5 directly, to produce a mixture of $\eta$ and $\eta'$ with the required property $h_m(\eta') = \kappa$ (clearly, Lemma 5 holds also for conditional entropies). ∎

We are in position to prove the second main theorem of this paper:

THEOREM 2: *Let $K$ be a compact metrizable Choquet simplex, and let $f$ be a non-negative affine function appearing as a non-decreasing limit of u.s.c. functions on $K$ ($f$ may attain infinity). Then there exists a minimal totally disconnected dynamical system $(Y, \sigma)$ being an almost 1-1 extension of $(G_{\overline{p}}, \tau)$, and an affine (onto) homeomorphism $\phi^*$: $K \mapsto K(Y, \sigma)$, such that for every $x \in K$, $f(x) = h(\phi^*(x))$, where $h$ denotes the entropy function on $K(Y, \sigma)$.*

*Proof:* It is known (as a consequence of the Lebesgue–Hausdorff theorem, see [K]) that every such function $f$ can be represented as the series

$$f = \sum_{j=1}^{\infty} f_j,$$

where each $f_j$ is affine non-negative and u.s.c. Let $\overline{\eta}_\infty$ be the vector Toeplitz function of Lemma 9. It will be convenient to enumerate the rows of $\overline{\eta}_\infty$ by $0, 1, 2, \dots$, so that the core appears in the zero row. Since this row has entropy zero, it can be ignored in the calculations of conditional entropies. As $\overline{\eta}_\infty$ is strictly ergodic, we have the following divergent series of numbers,

$$\sum_{j=1}^{\infty} h_j(\overline{\eta}^*_\infty).$$

By grouping the rows (if necessary) we can arrange that

$$h_j(\overline{\eta}^*_\infty) = \kappa_j$$

for each natural $j \geq 1$, where $\kappa_j$ is not smaller than the (finite) supremum of the function $f_j$. Each function $f_j$ is then approximated by a strictly decreasing sequence of affine continuous functions $f_{j,n}$, $(n \geq 0)$, $f_{1,0} \equiv \kappa_1$, $f_{j,0} < \kappa_j$ for $j \geq 2$.

The rest of the proof is similar to the proof of Theorem 1. We will use the same tags to denote analogous formulas as in that proof. We skip some details (definitions of some constants) which are identical as before. For each $n \geq 1$ we can assume that

$$f_{j,n-j+1} < f_{j,n-j} - \rho_{n-j} \quad \text{on } B_n, \tag{i}$$

for any $1 \leq j \leq n$.

*Step 1:* We pick some $\xi_0$, and we construct the map $\phi_1$ on $\operatorname{ex} B_1$ into $\overline{R}$-factors of $\overline{\eta}_\infty$ having for a given $e \in \operatorname{ex} B_1$ the following rows: the row indexed $-1$ containing the core, the row indexed 0 containing $\theta_e$ (this row is identical with the middle row in step 1 in the previous proof, and it has countably many discontinuities), and the rows $1, 2, \dots$ identical with the corresponding rows of $\overline{\eta}_\infty$. The extended map $\phi_1^*$ is an affine homeomorphism (between $B_1$ and its image). Moreover, $f_{1,0}(x) = h_1(\phi_1^*(x))$ and $f_{j,0}(x) < h_j(\phi_1^*(x))$ for each $j \geq 2$.

*Inductive assumption:* Suppose for some $n \geq 1$ we have defined the map $\phi_n$ from $\operatorname{ex} B_n$ into $\overline{R}$-factors of $\overline{\eta}_\infty$ having the core (in the first row indexed $-1$), and with at most countably many discontinuities in the row indexed 0. Moreover, the rows with indices larger than $n$ are the same as in $\overline{\eta}_\infty$ in every image $\phi_n(e)$. We assume that the extended map $\phi_n^*$ is an affine homeomorphism on $B_n$ satisfying

$$f_{j,n+1-j}(x) < h_j(\phi_n^*(x)) < f_{j,n-j}(x) + \rho_{n-j}, \tag{ii}$$

for every $1 \leq j \leq n$, and

$$f_{n+1,0} < h_{n+1}(\phi_n^*(x)) \tag{ii$'$}$$

(these conditions are fulfilled for $n = 1$). Also, we suppose that a number $\xi_{n-1}$ has been chosen.

*Step $n+1$:* We proceed as in the proof of Theorem 1 except that we modify the choice of $\xi_n$ by demanding (in addition to (iii)) that (iv) holds for $h_j(\mu)$, $f_{j,n-j}$ and $\rho_{n-j}$, for every $1 \leq j \leq n$ (now $\mu$ denotes any shift invariant measure over the countable product alphabet, hence $h_j(\mu)$ is well defined). An analogous modification concerns the choice of $\delta$; we demand that $|f_{j,n+1-j}(x) - f_{j,n+1-j}(y)| < \epsilon$ for each $1 \leq j \leq n$.

Then the construction goes unchanged until Lemma 6 is quoted. In this place to each $\overline{\eta}_i := \phi_n(e_i)$ we apply $n+1$ times our selective lowering entropy lemma

(Lemma 10). More precisely, notice that now each $\overline{\eta}_i$ is a vector Toeplitz function (its rows should be denoted by $\eta_{ij}$). We apply Lemma 10 to the vector Toeplitz functions $\eta_{i\{1,2,\dots,n+1\}}$ with the index $m$ ranging from 1 to $n+1$, and then we fill the remaining rows $-1, 0, n+2, n+3, \dots$ as they were in $\overline{\eta}_i$. In this manner new Toeplitz functions $\tilde{\eta}_i$ are constructed. Lowering of the entropy is done consecutively for the entropies $h_1, h_2, \dots, h_{n+1}$. In each step we use the inequality (ii) (or (ii′) in the last step). The parameter $\delta_h$ must be appropriately chosen so that the conditional entropies with larger indices still satisfy (ii) (or (ii′)) after modification. Recall that each step leaves the previously modified rows unchanged, so that we can set all conditional entropies precisely at the desired values:

$$h_j\left(\tilde{\eta}_i^*\right) = f_{j,n+1-j}(e_i),$$

$j = 1, 2, \dots, n+1$. Of course we can arrange that the distance between the measures $\overline{\eta}_i^*$ and $\tilde{\eta}_i^*$ is at most $\epsilon$. Observe that the conditional entropies $h_j(\tilde{\eta}_i^*)$ $(j \geq n+2)$ are not smaller than the corresponding values $h_j(\overline{\eta}_i^*)$, because we have replaced the first $n+1$ rows by their $\overline{R}$-factors, and we have not changed the further rows. This will ensure that (ii′) will hold in the next steps.

In the rest of the proof we introduce the following changes:

(vi) $$h_j(\tilde{\phi}_n^*(s)) = f_{j,n+1-j}(s) \quad \text{for } 1 \leq j \leq n+1,$$

(vii′) $$|h_j(\tilde{\phi}_n^*(s)) - h_j(\psi^*(s))| \leq \epsilon \quad \text{for } 1 \leq j \leq n+1.$$

The next change has only a formal character: to obtain an injective map we modify the row indexed 0 (it used to be the *middle* row in the previous proof), but the modification is exactly the same. From this point on the proof goes unchanged until the final estimates for entropy (three lines below the formula (x)). We write the same estimates but now they concern the functions $f_{j,n+1-j}$ and $h_j$ for $1 \leq j \leq n+1$. As a result, $h_j\left(\phi^*(x)\right) = f_j(x)$ for every $j \geq 1$ and $x \in K$.

Stages 3 and 4 of the proof are identical as in the previous proof, except that the resulting system $(Y, \sigma)$ is no longer expansive (but still totally disconnected) and it remains so after the application of Fact 1. ∎

In fact we have proved more than stated in Theorem 2:

THEOREM 3: *Let $K$ be a compact metrizable Choquet simplex, and let $(f_j)_{j\in\mathbb{N}}$ be a sequence of non-negative affine u.s.c. functions on $K$. Then there exists a minimal totally disconnected dynamical system $(Y,T)$ being an almost 1-1 extension of $(G_{\overline{p}}, \tau)$, in the form of an inverse limit of subshifts over some finite alphabets: $(Y,T) = \overleftarrow{\lim}_j(Y_j, \sigma)$, and an affine (onto) homeomorphism $\phi^*\colon K \mapsto$*

*$K(Y,T)$, such that $f_j$ coincides (via $\phi^*$) with the conditional entropy function $h(\mu_j|Y_{j-1})$, where $\mu_j$ is the projection of an invariant measure $\mu$ on $(Y,T)$ onto the factor $Y_j$.*

*Proof:* For complete proof of the above statement it is enough to observe that Fact 1 maintains the inverse limit representation of a totally disconnected system, i.e., its application to an inverse limit $(X,T) = \overleftarrow{\lim}_j(X_j,\sigma)$ produces a system $(Y,T)$ identical with the inverse limit $\overleftarrow{\lim}_j(Y_j,\sigma)$, where $(Y_j,\sigma)$ are the symbolic systems obtained via Fact 1 from $(X_j,\sigma)$. ∎

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